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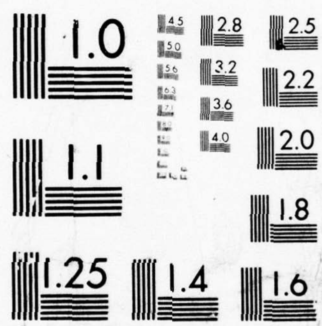
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PLASTICITY AND VARIABLE HEREDITY

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PLASTICITY AND VARIABLE HEREDITY

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ABSTRACT

Variable heredity is defined as a characteristic of advanced systems prevalently found in the living world that can permanently change their internal make-up due to the action of inputs and/or environment. Materials are special examples of such systems and "plasticity" results if mechanical inputs alone change the material properties.

For the operational definition of these characteristics a real system is idealized as a black box that receives inputs and emits outputs. Based on the comparison of suitable input-output pairs a definition of variable and invariable heredity is given. We distinguish between environment-induced variable heredity (example: aging) and that induced by inputs (example: "plasticity").

The operational definition of variable heredity is compared with that of rate-dependence and they are shown to be unrelated indicative that "fading memory" and "plasticity" do not necessarily exclude each other. The problems of characterizing materials with "plasticity" effects are discussed. Finally, the operational definition for "plasticity" obtained herein is applied as a necessary condition to various constitutive equations previously proposed for "plasticity". Some of these are shown to be unsuited for "plasticity". Owing to the special nature of the "plasticity" a functional constitutive equation showing history dependence in a mathematical sense may fail to reproduce "plasticity".

Introduction

For many years the theory of plasticity has occupied a special status in solid mechanics. It was almost always assumed that rate-independence, the yield surface and hardening laws are the only valid representation of plastic behavior. Indeed these notions were taken to be the distinguishing features of metal deformation behavior itself. At the same time this theory was not thought to be part of nonlinear continuum mechanics as it evolved during the last thirty years.

During the last ten years other approaches to plasticity were proposed which differ from the classical notions. Once one admits that the yield surface and other parts of classical plasticity theory are not the only mathematical expressions through which the phenomenon plasticity can be modelled one is immediately confronted with the question what are then the unique features of the body of knowledge known as plasticity. And if one has identified its unique features what are then the properties which distinguish mathematical expressions suitable for the description of plasticity from those that are not [1].

To many, dislocations are essential to plasticity. In a continuum theory state variables and separately postulated evolution laws are thought to be appropriate models. In other approaches an intrinsic time scale in the form of endochronic or of other theories appear important. For a long time the modelling of a yield point appeared to be essential and hypoelasticity theories were proposed. These are just a few of the possible approaches.

In this paper we propose to identify essential phenomena of "plasticity" by suitable thought experiments which are also easily performed in the laboratory. The basis of our identification rests with the well established fact by which every experienced technician can distinguish an annealed from a cold-

worked metal. He compares the stress-strain diagram of the two samples. The one with the highest stress-strain diagram is the cold-worked metal.

Although our proposed identification has the origin in this simple observation it is not limited to specific tests. Rather we use selectable input (stimuli)-output (response) pairs for the identification of the system of interest which is thought to be a black box.

Our aim is to identify the evolution of our system in time as it is subjected to a given constant environment and suitable inputs. After some technical preliminaries we come to the main result of the paper. We can precisely identify two basic classes of systems. The first class consists of systems with invariable heredity [2,3], i.e., systems that are always unchanged no matter what the environment and the stimuli. [Elasticity and viscoelasticity are members of this class.] The second class of systems has variable heredity. In these systems the environment and/or the inputs can change the <u>system properties permanently</u>. If these changes are caused by the environment alone (stimuli are absent) then we speak of aging. If the system properties are unaffected by the environment but may be changed by suitable stimuli we speak of stimulus induced variable heredity. The history dependence in the sense of plasticity [3] or simply "plasticity" is an example of a special kind of stimulus induced variable heredity. Aside from these hereditary properties rate-dependence and the aftereffect are identified for materials by suitable input-output experiments distinct from those used to identify the hereditary properties.

With these identifications we can establish necessary conditions which a mathematical model must fulfill if it is to represent a certain phenomenon. These conditions are very general but nevertheless permit a clear statement that some of the constitutive equations proposed previously for plasticity do

not exhibit variable heredity and are therefore, in our opinion, not a suitable model for plasticity.

Preliminaries

Our system of interest is idealized as a black box that receives inputs or stimuli or forcings $\underset{\sim}{\varphi}(\tau)$ on $[0,t]$. The inputs produce outputs or responses or response functions $\underset{\sim}{\rho}(t)$ at the present time t which depend on $\underset{\sim}{\varphi}(\tau)$ on $[0,t]$. For generality they are introduced as vector (tensor) quantities. The comparison of input-output pairs on some time interval is used to identify the phenomenon of interest. We must be able to compare outputs and to recognize differences between outputs. Therefore, we postulate the existence of a "primitive observer" [4] who will be able to discern whether two outputs for a given stimulus are identical or different. This assumes that our inputs and outputs have a defined zero value and that excursions from zero can be determined. Because of this simple requirement we can only identify very general properties.

We assume that there is a time $\tau = 0$ at which our process starts[1]. At this time we have as many identical samples of our system as we need. All the samples are in the same condition at $\tau = 0$, i.e., they have had the same method of preparation [5] and are subjected to a constant environment for $\tau \geq 0$. The only variables are time and the inputs which we select. The method is not limited to the identification of material properties, it can be applied to living systems as well[2] provided we have conditions which permit the identification of true system properties from the responses.

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[1] By stipulating this condition we deviate from treatments in continuum mechanics where it is generally assumed that we know the entire history from the distant past to the present time.

[2] An example would be a stem of bacteria in an incubator to which a chemical agent is administered and its effect on the bacteria is observed. Out black box would be identified with the stem of bacteria, the application of the chemical agent starting at $\tau = 0$ would be the stimulus. The increase (decrease) of the number of bacteria for $\tau > 0$ would be the response. We will return to this example later.

In the application of our methods to materials we must restrict ourselves to accelerationless, homogeneous motions (inputs) since we want to identify properties of constitutive equations [6]. In [2] we have shown that constant rate tensile testing, creep, relaxation and low-cycle fatigue loading constitute the best experimentally obtainable homogeneous motions in solids and therefore we should use inputs from this set of tests. They must be selected and can be based on the stress traction (displacement) vector. The output is then the displacement (stress traction) vector. By using $\underset{\sim}{\varphi}$ and $\underset{\sim}{\rho}$ we want to emphasize that kinematics (finite or infinitesimal motions) and the role of stress and strain are immaterial for our identification.

For the representation of our system we use a functional which is identified by a capital letter and which is thought to represent our system or material. We require that a zero input on [0,t] produces a zero output at t. Formally we have

$$\left.\begin{array}{l} \underset{\sim}{\rho}(t) = \underset{\sim}{K}\left(\underset{\sim}{\varphi}\left(\overset{t}{\underset{o}{\tau}}\right)\right)^{3)} \\ \text{with} \\ \underset{\sim}{K}\left(\underset{\sim}{0}\left(\overset{t}{\underset{o}{\tau}}\right)\right) = \underset{\sim}{0}, \end{array}\right\} \qquad (1)$$

where $\underset{\sim}{0}$ denotes the zero input or output. We do not assign any further properties to $\underset{\sim}{K}$ except it must be such that the conclusions valid for real systems can also be derived using its representation $\underset{\sim}{K}$.

For the representation of the phenomena of interest we propose sometimes separate functionals (identified by different capital letters) and then manipulate the arguments of them by using standard rules. We then require that one

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3) We mean by this symbolism simply that the present value of $\underset{\sim}{\rho} = \underset{\sim}{\rho}(\tau = t)$, $t \geq 0$ is determined by the function $\underset{\sim}{\varphi}(\tau)$ defined on [0,t]. From the information given in some of the examples an observer could also conclude $\underset{\sim}{\rho}(t) = \underset{\sim}{H}(\underset{\sim}{\varphi}(t))$, i.e., the present response $\underset{\sim}{\rho}$ is determined only by the present value of $\underset{\sim}{\varphi}$. The first conclusion is more general than the second and is therefore retained.

system should be describable by one functional. If we arrive at a contradiction by this procedure we must postulate certain other properties which are suitable for our purposes and which resolve the contradiction.

Invariable Heredity

Definition

Intuitively invariable heredity implies that the system does not change no matter what the stimulus or the environment. In addition, we have to consider the possibility that only certain stimuli leave the system unchanged. Therefore we differentiate between invariable hereditary response characterizing the latter possibility and invariable heredity for the characterization of the former. Formally we define in reference to Fig.1.

A system is said to have invariable heredity if for all $t = s$ and all $\underset{\sim}{\varphi}^{b}$

$$\underset{\sim}{\rho}(t)^{I} = \underset{\sim}{\rho}(s)^{II} = \underset{\sim}{\rho}(s)^{III} \tag{2}$$

no matter what $\underset{\sim}{\varphi}^{a}(\tau)$ and b.

We speak of invariable hereditary response if (2) is true for at least one b and at least one $\underset{\sim}{\varphi}^{a}(\tau)$. In the above, superscript Roman numerals define the response of the respective specimens in Fig.1.

Conditions on Constitutive Equations

The forcing histories for the three specimens used in Fig.1 are:

Specimen I

$$\underset{\sim}{\varphi}^{I} = \underset{\sim}{\varphi}^{b}(\tau)\,, \quad 0 \le \tau \le t, \quad \underset{\sim}{\varphi}^{b}(0) = \underset{\sim}{0} \tag{3}$$

Specimen II

$$\underset{\sim}{\varphi}^{II} = \left\{ \begin{array}{lll} \underset{\sim}{\varphi}^{a}(\tau)\,, & 0 \le \tau \le a\,; & \underset{\sim}{\varphi}^{a}(0) = \underset{\sim}{\varphi}^{a}(\tau \ge a) = \underset{\sim}{0} \\ \underset{\sim}{0} & a \le \tau \le b & \\ {}^{a}\underset{\sim}{\varphi}^{b}(\tau - b) & b \le \tau \le t & {}^{4)} \end{array} \right\} \tag{4}$$

4) By writing ${}^{a}\underset{\sim}{\varphi}^{b}$ we want to make clear that this input is preceded by $\underset{\sim}{\varphi}^{a}$; except for the shift on the timing axis ${}^{a}\underset{\sim}{\varphi}^{b}$ is identical to $\underset{\sim}{\varphi}^{b}$.

Specimen III

$$\underset{\sim}{\varphi}^{III} = \underset{\sim}{\varphi}^{b}(\tau - b), \quad b \le \tau \le t. \tag{5}$$

In the above $b > a > 0$.

Using (1) as a functional that describes our system the responses are computed to be

$$\underset{\sim}{\rho}^{I} = \underset{\sim}{K}\left(\underset{\sim}{\varphi}^{b}\overset{t}{\underset{o}{(\tau)}}\right) \tag{6}$$

$$\underset{\sim}{\rho}^{II} = \underset{\sim}{K}\left(\underset{\sim}{\varphi}^{a}\overset{a}{\underset{o}{(\tau)}} + {}^{a}\underset{\sim}{\varphi}^{b}\overset{t}{\underset{b}{(\tau-b)}}\right) - \underset{\sim}{K}\left(\underset{\sim}{\varphi}^{a}\overset{a}{\underset{o}{(\tau)}}\right), \tag{7}$$

and

$$\underset{\sim}{\rho}^{III} = \underset{\sim}{K}\left(\underset{\sim}{\varphi}^{b}\overset{t}{\underset{b}{(\tau-b)}}\right) . \tag{8}$$

For (2) to be true we must have for all $s = t$ using (6) and (8)

$$\underset{\sim}{K}\left(\underset{\sim}{\varphi}^{b}\overset{t}{\underset{b}{(\tau-b)}}\right) = \underset{\sim}{K}\left(\underset{\sim}{\varphi}^{b}\overset{s}{\underset{o}{(\eta)}}\right) = \underset{\sim}{K}\left(\underset{\sim}{\varphi}^{b}\overset{t}{\underset{o}{(\tau)}}\right) \tag{9}$$

and using (6) and (7)

$$\underset{\sim}{K}\left(\underset{\sim}{\varphi}^{a}\overset{a}{\underset{o}{(\tau)}} + {}^{a}\underset{\sim}{\varphi}^{b}\overset{t}{\underset{b}{(\tau-b)}}\right) = \underset{\sim}{K}\left(\underset{\sim}{\varphi}^{a}\overset{a}{\underset{o}{(\tau)}}\right) + \underset{\sim}{K}\left({}^{a}\underset{\sim}{\varphi}^{b}\overset{t}{\underset{b}{(\tau-b)}}\right) \tag{10a}$$

as well as

$$\underset{\sim}{K}\left({}^{a}\underset{\sim}{\varphi}^{b}\overset{t}{\underset{b}{(\tau-b)}}\right) = \underset{\sim}{K}\left({}^{a}\underset{\sim}{\varphi}^{b}\overset{s}{\underset{o}{(\eta)}}\right) = \underset{\sim}{K}\left(\underset{\sim}{\varphi}^{b}\overset{t}{\underset{o}{(\tau)}}\right) . \tag{10b}$$

Condition (9) is usually referred to as time origin translation invariance and (10a) has been called additivity under disjoint support [7]. We also list condition (10b) in which ${}^{a}\underset{\sim}{\varphi}^{b}(\eta)$ is the input following $\underset{\sim}{\varphi}^{a}(\tau)$. It involves first (9) and then requires that the response to a given identical stimulus must be unaltered by prior inputs. [Note that (10b) is not identical to (9).]

A constitutive equation represents invariable heredity if (9), (10a) and (10b) are true. In words these conditions represent

- invariance under time origin translation

and

- additivity under disjoint support
- no change in response due to prior forcings.

A constitutive equation represents invariable hereditary response if the above hold for all $\underset{\sim}{\varphi}^{b}$ and only for a specific prior exposure of b units of time and a specific prior loading $\underset{\sim}{\varphi}^{a}(\tau)$.

It should be noted that the response of specimen II is measured relative to the new origin introduced at $\eta = 0$. We have, therefore, eliminated the "permanent set" and do not attach any significance to this phenomenon[5].

Variable Heredity. Environment and Stimulus Induced

Following the definition of invariable heredity we now proceed to identify two kinds of variable heredity. For the first kind we observe that only the exposure to the environment changes the response. We then speak of environment induced variable heredity or simply aging. The other kind of variable heredity can be found in the absence of aging and is solely due to prior inputs. In this case we speak of stimulus induced variable heredity. These properties are now defined.

Environment Induced Variable Heredity (Aging)

Definition

Our system is said to exhibit aging if for some s = t

$$\underset{\sim}{\rho}^{I}(t) \neq \underset{\sim}{\rho}^{III}(s) \tag{11}$$

for all b and at least one $\underset{\sim}{\varphi}^{b}$.

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5) Our opinion deviates therefore from the notions of classical plasticity.

It exhibits aging response if (11) is true for at least one b and at least one $\underset{\sim}{\varphi}^{b}$ [6)].

Stimulus Induced Variable Heredity

Definition

In the absence of aging we speak of stimulus induced variable heredity if for some s = t and any b > a

$$\underset{\sim}{\rho}^{I}(t) \;\#\; \underset{\sim}{\rho}^{II}(s) \tag{12}$$

for all $\underset{\sim}{\varphi}^{a}$ and at least for one ${}^{a}\underset{\sim}{\varphi}^{b}$.

Our system exhibits a stimulus induced variable hereditary response if (12) is true for at least one $\underset{\sim}{\varphi}^{a}$ and at least one ${}^{a}\underset{\sim}{\varphi}^{b}$ [7)].

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6) Definition of aging for the example of footnote 2). Following Fig.1, Specimen I, we add a chemical at a certain rate (the stimulus $\underset{\sim}{\varphi}$) to the bacteria and observe the change in the number of bacteria with time (the response $\underset{\sim}{\rho}$).

On an identical sample of bacteria we repeat the above experiment after b units of time have elapsed, Specimen III of Fig.1. If the change in the number of bacteria is identical to that of the first experiment, then we can conclude that aging does not occur in our sample. If the outcome is different aging has occurred during b units of time. (To have a completely valid experiment, the size of the populations at $\tau = 0$ and $\tau = b$ should be identical.)

7) Stimulus induced variable heredity in the example of footnote 2). To simulate the conditions of the experiment with specimen II in Fig.1, we administer a chemical $\underset{\sim}{\varphi}^{a}(\tau)$ during [0,a], then we wait until changes in the population have ceased before we administer the test chemical (${}^{a}\underset{\sim}{\varphi}^{b}$) in the usual way. There are two possible outcomes:

1) The change in population is identical to that of the test with Specimen I.

2) The change in population is different from that of the test with Specimen II.

If outcome 2) is observed then we must necessarily conclude that the dose $\underset{\sim}{\varphi}^{a}$ given on [0,a] has changed the constitution of the bacteria, i.e., they may have developed an immune reaction (aging is assumed to be absent); we can now speak of stimulus induced variable heredity. [A completely valid test would require the same population size at $\tau = 0$ and $\tau = b$.]

Conditions on Constitutive Equations

Environment Induced Variable Heredity (Aging)

Comparison of (6), (8) and (11) shows the constitutive equation must be time origin translation variant for all b and at least one $\underset{\sim}{\varphi}^{b}$ to represent aging so that

$$\underset{\sim}{K}\left(\underset{\sim}{\varphi}^{b}\underset{b}{\overset{t}{(\tau - b)}}\right) \neq \underset{\sim}{K}\left(\underset{\sim}{\varphi}^{b}\underset{o}{\overset{t}{(\tau)}}\right). \tag{13}$$

To represent aging response we require time origin translation variance and therefore (13) to be true for at least one b and at least one $\underset{\sim}{\varphi}^{b}$.

Stimulus Induced Variable Heredity

Assuming time origin translation invariance, i.e., that (9) holds always, the constitutive equation must be able to represent nonadditivity under disjoint or change in response due to prior forcings. Therefore to represent stimulus induced variable heredity we must have

$$\underset{\sim}{K}\left(\underset{\sim}{\varphi}^{a}\underset{o}{\overset{a}{(\tau)}} + {}^{a}\underset{\sim}{\varphi}^{b}\underset{b}{\overset{t}{(\tau - b)}}\right) - \underset{\sim}{K}\left(\underset{\sim}{\varphi}^{a}\underset{o}{\overset{a}{(\tau)}}\right) \neq \underset{\sim}{K}\left(\underset{\sim}{\varphi}^{b}\underset{o}{\overset{t}{(\tau)}}\right) \tag{14}$$

which implies that either (10a) or (10b) or both are not true for all $\underset{\sim}{\varphi}^{a}$ and at least one ${}^{a}\underset{\sim}{\varphi}^{b}$.

To represent variable hereditary response the above conditions must hold for at least one $\underset{\sim}{\varphi}^{a}$ and at least one ${}^{a}\underset{\sim}{\varphi}^{b}$.

We take it as self evident that a constitutive equation must as a minimum be capable of satisfying the stated conditions if it is considered to be a valid representation of the so-defined system properties. Therefore, a constitutive equation for a system with environment-induced variable heredity must satisfy (13) and (10a) and (10b), a model for stimulus induced variable heredity must satisfy (14) and (9). A system with invariable heredity can only be represented by a mathematical expression that satisfies (9), (10a) and (10b).

The identifications and conditions were given very formally without regard to their physical implications which are far reaching. As an example the definition of stimulus induced variable heredity (test with specimens I and II in Fig.1) says that a form change of the response to the same stimulus is observed. A short reflection will show that this can only be possible physically *if the prior stimulus* $\underset{\sim}{\varphi}^a$ has permanently changed the internal make-up of our system. The systems ordinarily considered in mechanics (gases modelled by elastic balls, solids and fluids represented by springs and dashpots and combinations of then)[8] do not normally exhibit variable heredity. Indeed only Ref. [8] mentions variable heredity in connection with aging.

It appears that variable heredity is a property of advanced systems that can be encountered in materials and in the living world. Examples from the latter area are the immune reaction cited in the footnotes, the improving effects of exercise in athletics (the muscles strengthen due to prior stimuli), and relations between persons or groups of persons (attitude changes due to prior experience).

Before returning to the subject of plasticity it is important to mention that history dependence in a mathematical sense does not automatically represent variable heredity. The former is given by (1). Variable heredity, however, requires that either (13) or (14) or both hold, conditions which not every functional can fulfill.

Deformation Phenomena In Materials

Variable Heredity

The previous definitions of course apply to the deformation behavior of materials which can show only variable hereditary response as we may always find a $\underset{\sim}{\varphi}^a$ small enough (interpreted componentwise) such that referring to Fig.1 $\underset{\sim}{\rho}^I$ is indistinguishable from $\underset{\sim}{\rho}^{II}$. Also the environment may change the material.

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[8] A mere rearrangement of the constituents from one to another random orientation is not sufficient to cause variable heredity.

Indeed stimulus and environment induced variable hereditary response may occur simultaneously and they may interact at elevated temperature [3]. These synergistic effects will not be addressed in this paper where we treat the two kinds of variable heredity separately.

Specifically we consider the definition of variable hereditary response (12) as appropriate for the plasticity phenomenon, provided that the responses are only different in degree and not in kind as described in [3], p.64[9]. This type of variable hereditary response has been called history dependence in the sense of plasticity in [2,3]. We have given numerous examples which show that metals exhibit this phenomenon [2,3].

> We are therefore justified in defining "plasticity" as a special type of variable hereditary response. A constitutive equation suitable for "plasticity" must as a minimum satisfy (14) for at least one $\underline{\varphi}^{a}$ and one ${}^{a}\underline{\varphi}^{b}$. Equation (14) is therefore a necessary condition for the modelling of "plasticity".

This operational definition of "plasticity" has a number of consequences which are briefly discussed.

A constitutive equation showing history dependence in a mathematical sense does not necessarily represent "plasticity. This statement is clear in light of the previous discussion.

The above definition excludes the model of an elastic perfectly plastic plastic material from "plasticity" [9]. This fact is not disturbing since the special nature of this model has long been recognized (no growth law for the yield surface is necessary in this case).

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9) Living systems may exhibit responses which are different in kind.

It is impossible to infer "plasticity" from one test alone. Only the comparison of two tests (specimens I and II) permit the identification. So a given constitutive equation may very well match a stress-strain diagram, or a set of creep curves perfectly without reproducing "plasticity". The critical test for a constitutive equation is therefore its behavior during loading, unloading and subsequent reloading.

We know that the physical reason for the observed history dependence in the sense of plasticity rests with the possible permanent microstructural changes induced in materials by deformation. (The material at $\tau = a$ can be different from the material at $\tau = 0$, Specimen II in Fig.1; in metals the dislocation density at the two points may be different by several orders of magnitude.) These changes proceed during deformation. Macroscopically we can only note the difference between $\underset{\sim}{\rho}^{I}$ and $\underset{\sim}{\rho}^{II}$ and we can only use this difference as a criterion for "plasticity". Consequently we must accept a constitutive equation for "plasticity" as long as it can represent such a difference.

The definition of "plasticity" is very broad and only necessary. Only very general properties are delineated. We are not in a position to give necessary and sufficient conditions for the representation of "plasticity". In this case we would have to distinguish between various ranges of temperature, large or small deformations, types of environments and classes of materials.

<u>Rate-Dependence, Rate-Independence, Aftereffect and Fading Memory</u>

For completeness we are including the operational definition of the first three phenomena. These definitions can again be used to give necessary conditions on constitutive equations for the representation of these phenomena. They are shown to be distinct from the ones related to variable heredity.

Rate-Dependence, Rate-Independence

Definition

It is easiest to consider a nondecreasing forcing $\underset{\sim}{\varphi}(\tau)$ with $\underset{\sim}{\varphi} \cdot d\underset{\sim}{\varphi} \geq 0$ and an accelerated (retarded) forcing $\underset{\sim}{\varphi}(\alpha\tau)$ with $\alpha > 0$. The forcing $\underset{\sim}{\varphi}(\tau)$ and the accelerated forcing $\underset{\sim}{\varphi}(\alpha\tau)$ reach the same value at $\tau = t$ and $\tau = t/\alpha$, respectively. We speak of rate-independence if

$$\underset{\sim}{\rho}(t) = \underset{\sim}{\rho}(t/\alpha) \tag{15}$$

is true for all t and all α (see also the definition in [10]). If (15) is not true for some t or some α we speak of rate-dependence.

A graph of these conditions for linear $\underset{\sim}{\varphi}$ is given in Fig.2. These conditions can easily be performed in experiments.

Condition on Constitutive Equations

Using (1) and (15) a necessary condition for the representation of rate-independence is

$$\underset{\sim}{K}\left(\underset{\sim}{\varphi}\overset{t}{\underset{o}{(\tau)}}\right) = \underset{\sim}{K}\left(\underset{\sim}{\varphi}\overset{t/\alpha}{\underset{o}{(\alpha\tau)}}\right) \;^{10)} \tag{16}$$

for all t and α. If (16) does not hold for some t and α we speak of rate dependence.

Aftereffect

It is generally known that after a stimulus (loading) has returned to zero the absolute value of the response gradually decreases to some constant value for a majority of materials. Some materials attain the constant value

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[10] Unfortunately this notation is not completely clear. It is not meant to imply that equality can always be obtained by introducing a new time variable $\eta = \alpha\tau$. A notation like $\underset{\sim}{K}(\underset{\sim}{\varphi}\overset{t}{\underset{o}{(\tau)}}, t-\tau)$ is less ambiguous but less used than $\underset{\sim}{K}(\underset{\sim}{\varphi}\overset{t}{\underset{o}{(\tau)}})$.

immediately after the stimulus has returned to zero, they therefore exhibit no aftereffect. (We have on purpose avoided the term recovery which sometimes but not always means aftereffect.)

Definition

In reference to Fig.1, specimen II, we define for ${}^{a}\underset{\sim}{\varphi}^{b} = \underset{\sim}{0}$, the aftereffect operationally

$$\|\underset{\sim}{\rho}(t_1)\| \geq \|\underset{\sim}{\rho}(t_2)\| \tag{17}$$

for every $t_2 \geq t_1 \geq a$. For no aftereffect the equality holds always. The notation $\|\underset{\sim}{\rho}\|$ designates the absolute value of each component of $\underset{\sim}{\rho}$.

Condition on Constitutive Equations

From (1) and (17) we obtain a necessary condition for the representation of the aftereffect. It is

$$\|\underset{\sim}{K}\Big(\underset{\sim}{\varphi}^{a}\overset{a}{\underset{o}{(\tau)}}\Big)\|_{\tau=t_1} \geq \|\underset{\sim}{K}\Big(\underset{\sim}{\varphi}^{a}\overset{a}{\underset{o}{(\tau)}}\Big)\|_{\tau=t_2} \tag{18}$$

for all $t_2 \geq t_1 \geq a$. The absence of an aftereffect requires the equality sign in (18). Again the notation proposed in footnote 10) would be advantageous since from this notation we could immediately see that rate-independence implies no aftereffect and vice versa.

Using (16) and the notation of footnote 10) we obtain

$$\underset{\sim}{K}\Big(\underset{\sim}{\varphi}(\tau)\overset{t}{\underset{o}{,}}\ t-\tau\Big) = \underset{\sim}{K}\Big(\underset{\sim}{\varphi}(\eta)\overset{t}{\underset{o}{,}}\ \frac{t-\eta}{\alpha}\Big) \tag{19}$$

which must be true for arbitrary α to represent rate-independence. Therefore $\underset{\sim}{K}$ cannot depend on $t-\tau$. This fact immediately gives the equality in (18). Starting with (18) and requiring no aftereffect we can show that $\underset{\sim}{K}$ cannot depend on t which in turn insures rate-independence.

We see now that rate-dependence and the aftereffect are related. Further the modelling of the aftereffect imposes the restrictions (18) on the constitutive functionals. These restrictions are identical to those imposed on viscoelastic materials [11]. They are shown to be related [11] to the "fading memory hypothesis" imposed in the theory [6] of constitutive functionals.

Representation and Characterization of Materials

The rate-dependence, the aftereffect and fading memory are related to each other. They affect those properties of materials which were called viscous or rheological in [5,12,17]. Physically the details of the deformation mechanisms, i.e., friction at the present time are responsible for the presence or absence of viscosity. Mathematically viscosity involves condition (16).

Variable heredity, on the other hand, involves conditions (13) and (14) which are mathematically separate from (16). Physically variable heredity is caused by the accumulated effects of past exposure to environment or to inputs. In the last case this phenomenon is sometimes referred to as mechanical aging. These accumulated effects are called structural mechanisms in [5,12,17] and are separate from viscosity.

For illustration we have listed the physical phenomena and the necessary mathematical conditions in Table 1 to illustrate our point. The phenomena variable heredity and rate-dependence (aftereffect) are unrelated. Continuum mechanics theory has mostly concentrated on the fading memory aspects and has therefore almost exclusively dealt with the viscosity of deformation behavior. Variable heredity has not yet been generally recognized as an important phenomenon in the evolution in time of material systems. This may be the reason why "plasticity" is not yet included in these theories.

Based on the evidence presented so far it would seem natural to use separate repositories for variable heredity and viscosity (rate-dependence). This approach was followed in [3,5,12,15,17].

Since we have defined two kinds of variable heredity it is natural to ask how these two phenomena can be characterized (i.e., what kind of tests are necessary to obtain the material properties) and how one might represent them in constitutive equations.

To characterize an aging system one could vary b for a fixed $\underset{\sim}{\varphi}^b$ [see Eqs. (5) and (8)] and them repeat the process for a different $\underset{\sim}{\varphi}^b$ until one knows how b and $\underset{\sim}{\varphi}^b$ modify the response. In reality a different specimen is necessary for each test.

For the representation of aging system an explicit dependence on time of the constitutive equation if frequently used, i.e.

$$\underset{\sim}{\rho}(t) = \underset{\sim}{K}\left(\underset{\sim}{\varphi}\binom{t}{\tau}_{o}, t\right) . \tag{20}$$

The response of specimen I for a material represented by (20) is

$$\underset{\sim}{\rho}^{I}(t) = \underset{\sim}{K}\left(\underset{\sim}{\varphi}^{b}\binom{t}{\tau}_{o}, t\right) \tag{21}$$

and of specimen III

$$\underset{\sim}{\rho}^{III}(s) = \underset{\sim}{K}\left(\underset{\sim}{\varphi}^{b}\binom{s}{\eta}_{o}, s+b\right) \tag{22}$$

and we see that (11) is satisfied.

The representation (20) is said to violate the principle of material indifference [13, footnote p.45]. There are at least two ways of reconciling this potential conflict. One way is to adopt the derivation in [14], Eqs. (2.11) or (2.12). Another way is to consider that the present response is a functional of both the mechanical and the environmental input. Since the

latter is constant here its functional dependence can be "integrated out" leaving only a function of time and a functional of the mechanical input, i.e., Eq. (20), [15]. If we therefore interpret (20) not as a fundamental form but rather a specific representation valid for constant environment only no conflict arises.

For the characterization of stimulus induced variable heredity we must vary the $\underset{\sim}{\varphi}^{a}$ for a given fixed ${}^{a}\underset{\sim}{\varphi}^{b}$ and then repeat the process for a different fixed ${}^{a}\underset{\sim}{\varphi}^{b}$ until we know how the $\underset{\sim}{\varphi}^{a}$ change the response. However, there are difficulties.

Strictly speaking this is a formidable, if not impossible task as one may have to run all conceivable combinations of $\underset{\sim}{\varphi}^{a}$ and ${}^{a}\underset{\sim}{\varphi}^{b}$. Elsewhere this difficulty has been recognized as evidenced by the statement: "In a strict sense it is not possible to predict the strain components which will be found for a given stress *history; the experiment* itself must be run to get the answer" [16].

However, the situation is not quite as complex as it appears when we deal with history dependence in the sense of plasticity and with metals. Their responses retain certain characteristics which are invariant with respect to prior deformation [3, pp.63-66]. Further there are certain stimuli, notably the "elastic ones" which do not appear to cause "plasticity" effects. It may therefore be possible to characterize the history dependence of metals with a limited number of tests and suitable interpolations (see the discussion by E.H. Lee and E. Kröner on p.86 of [16]).

The above definition of history dependence was given for a continuum. Nothing in that definition suggests that a theory of simple materials (in the sense of [6]) would not be capable of reproducing history dependence in the sense of plasticity. Therefore the knowledge of the response to macroscopically homogeneous motions permits the predictions under macroscopically inhomogeneous

conditions. Although metals may be "nonsimple bodies" on a microscopic level [16,p.47] it appears that they can be represented macroscopically within the continuum theory of simple materials as defined in [6].

Suppose two primitive observers are told that they will have to compare the outcome of the tests on the same material with specimen I and II, respectively (see Fig.1). Observer I witnesses the tests with specimens I and II, Observer II sees <u>only</u> the second part ($0 \leq \eta \leq s$) of the test on specimen II. We stipulate that (12) holds, i.e., we have stimulus induced variable heredity.

Since they know that the same material is tested both observers would use the same functional for representing the data. However, because of (12) (we use $\underset{\sim}{\varphi}^{b}$ instead of ${}^{a}\underset{\sim}{\varphi}^{b}$ since Observer II does not know that $\underset{\sim}{\varphi}^{a}$ was applied)

$$\underset{\sim}{K}\left(\underset{\sim}{\varphi}^{b}\,\overset{s}{\underset{o}{(\eta)}}\right) \neq \underset{\sim}{K}\left(\underset{\sim}{\varphi}^{b}\,\overset{t}{\underset{o}{(\tau)}}\right) \tag{23}$$

for some $s = t$, a result which contradicts our initial stipulation that one functional can describe a material.

Observer I must necessarily conclude that $\underset{\sim}{\varphi}^{a}$ on $0 \leq \tau \leq a$ must have changed the material from $\underset{\sim}{K}$ to say $\hat{\underset{\sim}{K}}$. However, there must be a way to obtain $\hat{\underset{\sim}{K}}$ from $\underset{\sim}{K}$. Indeed one can write

$$\underset{\sim}{K}\left(\underset{\sim}{\varphi}^{a}\,\overset{a}{\underset{o}{(\tau)}} + {}^{a}\underset{\sim}{\varphi}^{b}\,\overset{t}{\underset{b}{(\tau-b)}}\right) - \underset{\sim}{K}\left(\underset{\sim}{\varphi}^{a}\,\overset{a}{\underset{o}{(\tau)}}\right) = \hat{\underset{\sim}{K}}\left(\underset{\sim}{\varphi}^{b}\,\overset{s}{\underset{o}{(\eta)}}\right) \tag{24}$$

where we have again used $\underset{\sim}{\varphi}^{b}$ instead of ${}^{a}\underset{\sim}{\varphi}^{b}$ on the right-hand side to demonstrate that we do not know the forcings for $\eta \leq 0$ if we use $\hat{\underset{\sim}{K}}$ as a representation.

The above can be related to commonly accepted notions in materials science. "We regard it as self-evident, then, all current properties of a material are entirely determined by its current state" [18].

The "current state" is represented in (24) by the "material constants" in $\underset{\sim}{K}$ and $\hat{\underset{\sim}{K}}$, respectively. Since the input is identical in both tests the "material constants" in $\underset{\sim}{K}$ and $\hat{\underset{\sim}{K}}$ must be different to represent the different states.

The task in materials science is to determine the current structure knowing the structure at some previous time and what happened to the materials between now and the previous time.

The task in continuum mechanics is similar. Equation (24) says that knowing $\underset{\sim}{K}$, i.e., the material constants, at $\tau = 0$ and the forcings on $[0,b]$ should enable us to determine $\hat{\underset{\sim}{K}}$ (the material constants at $\tau = b$). (It is of course open how $\hat{\underset{\sim}{K}}$ and $\underset{\sim}{K}$ must be constructed such that this determination is possible.)

It is interesting to note that similar ideas were expressed in [5, p.125]: "To determine the actual thermomechanical state .... it is insufficient to have the actual deformation temperature configuration of a particle $\underset{\sim}{X}$ but we additionally need the method of preparation of this configuration". In the language of materials science method of preparation can be interpreted as current state.

Once the need for information on the current state or the method of preparation is recognized one must immediately ask whether all states are equivalent or whether there is a preferred state.

From experience it appears that the annealed state (all the effects of prior mechanical loading have been removed by appropriate heat treatment) is a preferred state. A material can stay in this state if the inputs are small but no mechanical stimulus can return the material to this state once it has departed from it.

If the annealed state was left behind by a suitable stimulus then we can always define a new state relative to which we must characterize the material. The part of the stimulus leading from the annealed state to a new state that

caused "plasticity" will together with the old state be absorbed in the new method of preparation or in the new state of the material. As such we can go from one state to the other as implied by (24). Then the initial annealed state does not appear anymore in an explicit way.

Application of the Necessary Conditions Developed Earlier

Here we select from various constitutive equations considered at one time or another for plasticity. We check whether (14) holds. Only if the answer is affirmative do we consider it a valid representation for "plasticity". (Of course there may be other objections to models found to be valid by this procedure.)

Classical Plasticity and Classical Viscoplasticity

Strains are split either additively, e.g. [19] and others or multiplicatively, e.g. [20] and others, and a growth law for the yield surface must be given such that the elastic range can change under the application of at least one $\underset{\sim}{\varphi}^a$ in Fig.1. The change in the elastic range alone is sufficient that (14) can hold and therefore most classical theories can represent stimulus induced variable heredity.

Linear and Nonlinear Viscoelasticity

Integral representations of the form

$$\underset{\sim}{\rho}(t) = \int_0^t \underset{\approx}{J}(t-\tau) \cdot \frac{d\underset{\sim}{\varphi}}{d\tau} + \int_0^t \int_0^t \underset{\approx}{G}(t-\tau_1, t-\tau_2) \cdot \frac{d\underset{\sim}{\varphi}}{d\tau_1} \cdot \frac{d\underset{\sim}{\varphi}}{d\tau_2} d\tau_1 d\tau_2 + \ldots. \quad (25a)$$

or

$$\underset{\sim}{\rho}(t) = \int_0^t \underset{\approx}{H}\Big(t-\tau, \varphi(\tau)\Big) \cdot \frac{d\underset{\sim}{\varphi}}{d\tau} d\tau \quad (25b)$$

or

$$\underset{\sim}{\rho}(t) = \underset{\sim}{\varphi}(t) + \int_0^t \underset{\approx}{K}(t-\tau) \cdot \underset{\sim}{f}\Big(\underset{\sim}{\varphi}(\tau)\Big) d\tau \quad (25c)$$

with $\underset{\sim}{f}(\underset{\sim}{0}) = \underset{\sim}{0}$ have the property required in (1) and can be written in the form

$$\underset{\sim}{\rho}(t) = \underset{\sim}{K}\left(\underset{\sim}{\varphi}(\tau), \overset{t}{\underset{o}{\,}} t-\tau\right) . \tag{25}$$

They all show invariable heredity and are unsuitable to model "plasticity"[11].

Intrinsic Time, Endochronic Theory, Arc Length Parametrization

These theories postulate the existence of an internal time ([21] - [30] and earlier papers quoted therein) which is mostly based on the second invariant of strain increments. The theories can be formulated for infinitesimal and finite strains and a stress based time scale has also been proposed [29].

Most theories employ convolution integrals in the intrinsic time scale z and can be written symbolically in a form similar to (25)

$$\underset{\sim}{\rho}\left(z(t)\right) = \underset{\sim}{F}\left(\underset{\sim}{\varphi}(z'), \overset{z(t)}{\underset{o}{\,}} z-z'\right) . \tag{26}$$

Equation (26) exhibits additivity under disjoint support so that (10a) holds. The modelling of variable heredity rests with violation of (9) or (10a) depending on the relation between the intrinsic and real time, see Appendix I.

Appendix I clearly demonstrates that the introduction of convolution integrals in intrinsic time $\hat{z}$ with $\hat{z}$ defined as

$$\hat{z}(t) = \int_{o}^{t} (d\underset{\sim}{\varphi} \cdot \underset{\sim}{P} \cdot d\underset{\sim}{\varphi})^{\frac{1}{2}} \, d\tau \tag{27}$$

is not sufficient for the modelling of stimulus induced variable heredity[12].

---

11) If the matching of metal stress-strain or creep curves were to be criterion for "plasticity, then every one of these equations could be a valid model.

12) Again if only the matching of tensile and shear stress-strain diagrams is considered to be important to represent "plasticity", then an intrinsic time alone is sufficient.

An additional mapping between $\hat{z}$ and the z must be employed in (26). This important ingredient is normally not considered a part of intrinsic time theories. It is, except for the condition $\frac{dz}{d\hat{z}} > 0$, unrestricted from a theoretical point of view.

If this additional mapping is not used and the theory is formulated only as a convolution integral in the $\hat{z}$-parameter, then it fails to reproduce "plasticity". An example in case is the theory presented in [24].

If the convolution form in (26) is not used, e.g., if the $z - z'$ dependence is replaced by a z, z' dependence then (27) alone is sufficient to model plasticity. In this case g in (A5) and (A15) has to be annulled to avoid aging in real time.

Hypoelasticity

The hypoelastic relations

$$\overset{\circ}{\underset{\sim}{T}} = \underset{\approx}{\pounds}(\underset{\sim}{T}) \cdot \underset{\sim}{D} \tag{28}$$

have been frequently considered as a "plasticity" model [31 - 34]. In the above $\underset{\sim}{T}$ denotes the Cauchy stress tensor, $\overset{\circ}{\underset{\sim}{T}}$ a spin invariant derivative and $\underset{\sim}{D}$ the rate of deformation tensor.

This equation was considered as a "plasticity" model because of its rate-independence and its ability of reproducing stress-strain diagrams exhibiting yielding [33], also mathematical expressions can be obtained from (28) reminiscent of yield surfaces [32].

In [30] relations of the type (28) were refuted on the grounds that yielding occurs in reality [31] at strains "justifiably considered small", whereas (28) can only predict yield-type phenomena at large strains. However, it was shown in [33] that large strains are not a prerequisite for the prediction of yielding if a special form of (28) is chosen (see the discussion on p.17 of [33]). Also it is remarked in [34] that (28) has no relation to the plasticity phenomenon.

We refute (28) on the basis of its properties as they evolve in time. "Plasticity" requires the modelling of stimulus induced variable heredity. Equation (28), however, represents invariable heredity, as shown in Appendix II.

Postscript: Initial thoughts on this subject were given in [35]. Since the completion of the first draft of this paper (Spring 1976) the phenomenon called here stimulus induced variable heredity has been used to formulate a theory of material divagation on the basis of continuum mechanics [36]. Also an ideal material is defined in [36] with properties similar to our material with invariable heredity. It is a pleasure to acknowledge interesting discussions with Professor D.C. Stouffer, University of Cincinnati, which have contributed to this final version. Some thoughts on damage accumulation are given in [37].

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TABLE 1

VARIOUS PHENOMENA AND NECESSARY CONDITIONS ON FUNCTIONALS INTENDED FOR THEIR REPRESENTATION

Necessary Condition \ Name	Invariable Heredity*	Stimulus Induced Variable Heredity*	Environment Induced Variable Heredity*	Rate-Dependence	Rate-Independence
$\underset{\sim}{K}\left(\underset{\sim}{\varphi}\big(\overset{a}{\underset{o}{\tau}}\big) + {}^{a}\underset{\sim}{\varphi}^{b}\big(\overset{t}{\underset{b}{\tau - b}}\big)\right) = \underset{\sim}{K}\left(\underset{\sim}{\varphi}\big(\overset{a}{\underset{o}{\tau}}\big)\right) + \underset{\sim}{K}\left({}^{a}\underset{\sim}{\varphi}^{b}\big(\overset{t}{\underset{b}{\tau - b}}\big)\right)$ $\underset{\sim}{K}\left({}^{a}\underset{\sim}{\varphi}^{b}\big(\overset{s}{\underset{o}{\eta}}\big)\right) \underset{s=t}{=} \underset{\sim}{K}\left(\underset{\sim}{\varphi}^{b}\big(\overset{t}{\underset{o}{\tau}}\big)\right)$	True	Not True for some s = t	True	Unrelated	Unrelated
$\underset{\sim}{K}\left(\underset{\sim}{\varphi}^{b}\big(\overset{t}{\underset{b}{\tau}}\big)\right) \underset{s=t}{=} \underset{\sim}{K}\left(\underset{\sim}{\varphi}^{b}\big(\overset{s}{\underset{o}{\eta}}\big)\right)$	True	True	Not True for some s = t	Unrelated	Unrelated
$\underset{\sim}{K}\left(\underset{\sim}{\varphi}\big(\overset{t}{\underset{o}{\tau}}\big)\right) = \underset{\sim}{K}\left(\underset{\sim}{\varphi}\big(\overset{t/\alpha}{\underset{o}{\alpha\tau}}\big)\right)$	Unrelated	Unrelated	Unrelated	Not always true	Always true

* For simplicity no distinction is made between property and response. Also interactions between stimuli and environment induced effects are excluded.

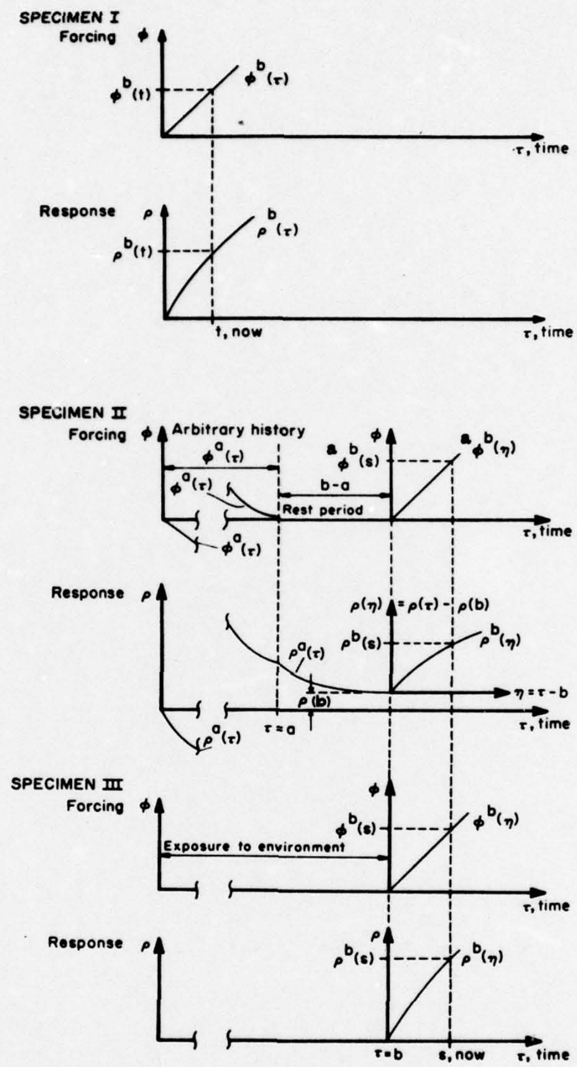

Fig.1. Schematic illustrating the operational definition of variable heredity. At $\tau = 0$, when they are exposed to the test environment, all specimens are supposed to have the same initial properties. Note, that for Specimens II and III a new origin is introduced at $\tau = b$.

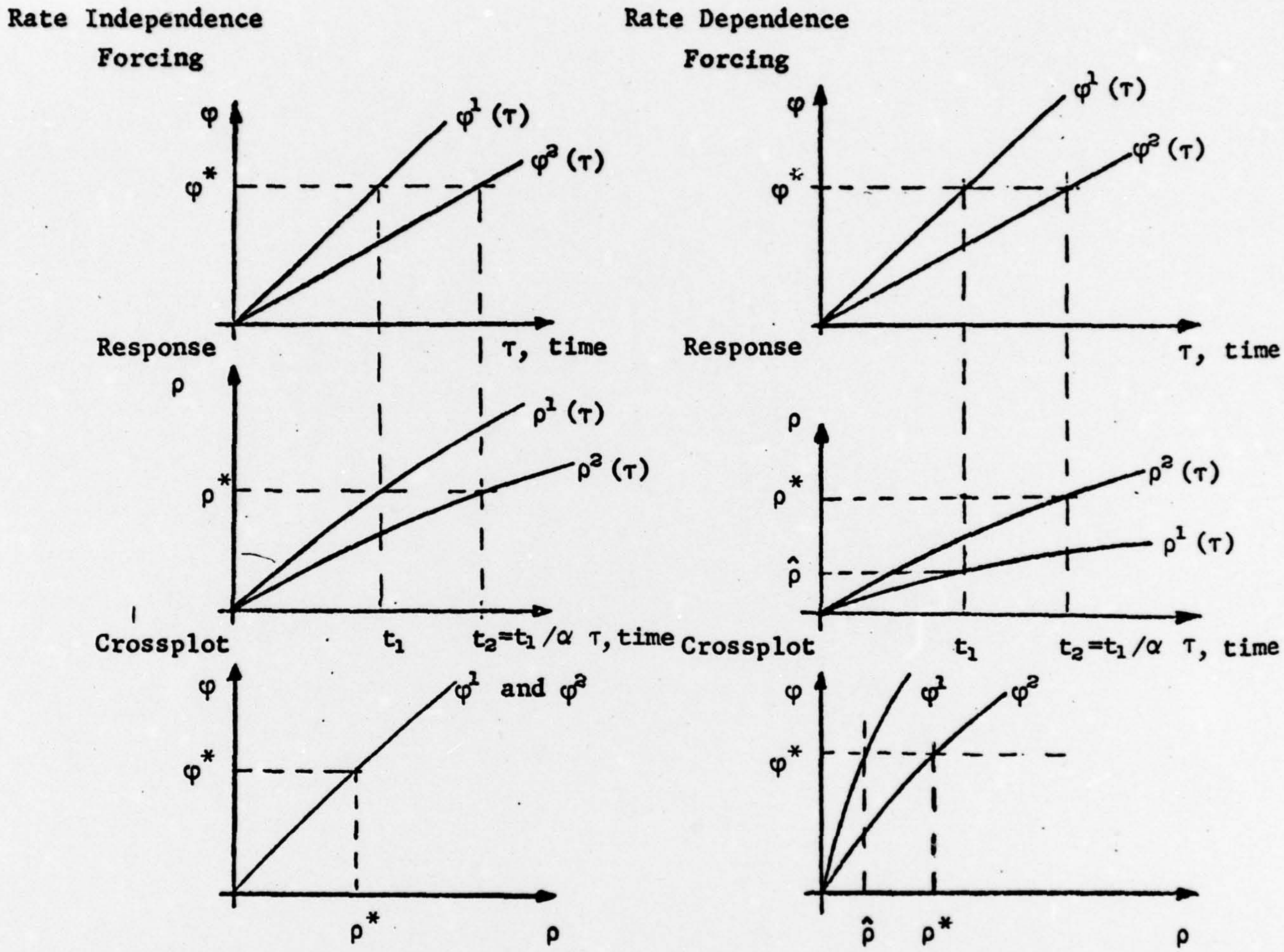

Figure 2. Definition of Rate Dependence and Rate Independence on the Basis of Linear Forcing Functions.

APPENDIX I

We subject (26) to (4) and obtain considering (1)

$$\underset{\sim}{\rho}(z_t) = \underset{\sim}{F}\left(\underset{\sim}{\varphi}^{a}\left(\underset{o}{\overset{z_a}{z'}}\right),\ z - z'\right) + \underset{\sim}{F}\left({}^{a}\underset{\sim}{\varphi}^{b}\left(\underset{z_b}{\overset{z_t}{z'}} - z_b\right),\ z - z'\right) \tag{A1}$$

where $z_t = z(\tau = t) = z$, $z_a = z(\tau = a)$ and $z_b = z(\tau = b)$. The second term can be rewritten by introducing $z'' = z' - z_b$ to yield

$$\underset{\sim}{F}\left({}^{a}\underset{\sim}{\varphi}^{b}\left(\underset{z_b}{\overset{z_t}{z'}}\right),\ z - z'\right) = \underset{\sim}{F}\left({}^{a}\underset{\sim}{\varphi}^{b}\left(\underset{o}{\overset{z_t - z_b}{z''}}\right),\ z_t - z_b - z''\right),$$

so that

$$\underset{\sim}{\rho}(z_s) = \underset{\sim}{F}\left({}^{a}\underset{\sim}{\varphi}^{b}\left(\underset{o}{\overset{z_s}{z''}}\right),\ z_s - z'\right) \tag{A2}$$

where we have set $z_s = z_t - z_b$. On the other hand for specimen I of Fig.1

$$\underset{\sim}{\rho}(z) = \underset{\sim}{F}\left(\underset{\sim}{\varphi}^{b}\left(\underset{o}{\overset{z}{z'}}\right),\ z - z'\right). \tag{A3}$$

Since $\underset{\sim}{\varphi}^{b}$ is identical to ${}^{a}\underset{\sim}{\varphi}^{b}$ in (A2) and (A3) we see that the two responses will be equal if the intrinsic times z_s and z are equal in both cases. Formally, an intrinsic time formulation will <u>exhibit invariable heredity</u> if

$$z = z_s \quad \text{for} \quad t = s$$

where

$$\left.\begin{aligned} z &= z(t) \text{ of specimen I} \\ \text{and} \quad z_s &= z(t) - z(b) \text{ of specimen II}. \end{aligned}\right\} \tag{A4}$$

It should be stressed that (A4) is valid for single as well as multiple convolution integral constitutive equations.

Before we proceed further we want to remark that the use of convolution integrals in intrinsic time immediately implies additivity under disjoint support so that the repository for history dependence in the sense of plasticity rests entirely with the intrinsic time z [see Eq. (A4)]. We now proceed to investigate the properties of some intrinsic time scales.

In [29] the following intrinsic time z is postulated, see p.859 and 860

$$dz = \varkappa[d\zeta^2 + g^2 d\tau^2]^{\frac{1}{2}} \tag{A5}$$

with

$$d\zeta = \frac{d\xi}{f(\xi)} \tag{A6}$$

and

$$d\xi = (\dot{\underset{\sim}{E}} \cdot \underset{\approx}{P} \cdot \dot{\underset{\sim}{E}})^{\frac{1}{2}}\, d\tau = (\dot{\underset{\sim}{\varphi}} \cdot \underset{\approx}{P} \cdot \dot{\underset{\sim}{\varphi}})^{\frac{1}{2}}\, d\tau \tag{A7}$$

where $\dot{\underset{\sim}{E}}$ is the material derivative of the finite strain tensor and $\underset{\approx}{P}$ is a positive definite tensor which may depend on $\underset{\sim}{E}$. Since also a stress-based time scale is proposed in [29] we have introduced our forcing function tensor $\underset{\sim}{\varphi}$ to cover both cases. Integration of (A5) between z_t and z_b yields

$$z_t - z_b = \int_b^t \varkappa\left[\left(\frac{d\xi(\tau)}{d\tau}\,\frac{1}{f(\xi(\tau))}\right)^2 + g^2\right]^{\frac{1}{2}} d\tau \tag{A8}$$

with $\tau' = \tau - b$ the above can be transformed into

$$z_t - z_b = \int_0^{t-b} \varkappa\left[\left(\frac{d\xi(\tau'+b)}{d\tau'}\,\frac{1}{f(\xi(\tau'+b))}\right)^2 + g^2\right]^{\frac{1}{2}} d\tau' . \tag{A9}$$

Now

$$\xi(\tau'+b) = \int_0^a [\dot{\underset{\sim}{\varphi}}^a \cdot \underset{\approx}{P} \cdot \dot{\underset{\sim}{\varphi}}^a]^{\frac{1}{2}}\, d\tau + \int_0^{\tau'} [\dot{\underset{\sim}{\varphi}}^b \cdot \underset{\approx}{P} \cdot \dot{\underset{\sim}{\varphi}}^b]^{\frac{1}{2}}\, d\tau = \xi(a) + \xi(\tau') \tag{A10}$$

and

$$d\xi(\tau'+b) = [\dot{\varphi}^b \cdot P \cdot \dot{\varphi}^b]^{\frac{1}{2}}\, d\tau' = d\xi(\tau') . \tag{A11}$$

Substitution of (A10) and (A11) into (A9) yields

$$z_s = z_t - z_b = \int_0^{t-b=s} \varkappa\left[\left(\frac{d\xi(\tau')}{d\tau'}\,\frac{1}{f(\xi(a)+\xi(\tau'))}\right)^2 + g^2\right]^{\frac{1}{2}} d\tau' . \tag{A12}$$

On the other hand we obtain for z in (A3)

$$z = \int_0^t \varkappa\left[\left(\frac{d\xi(\tau)}{d\tau}\,\frac{1}{f(\xi(\tau))}\right)^2 + g^2\right]^{\frac{1}{2}} d\tau . \tag{A13}$$

A comparison of (A12) and (A13) shows that $z_s = z$ for $t = s$ if

$$f(\xi(a) + \xi(\tau)) = f(\xi(\tau)), \tag{A14}$$

since the name of the dummy variable is immaterial. Condition (A14) is certainly true if f is a constant.

The same procedure can be repeated for the "old endochronic" time proposed in [22,23] where

with

$$\left. \begin{aligned} dz &= \frac{1}{f(\zeta)} \frac{d\zeta}{d\tau} d\tau \\ d\zeta &= [d\underset{\sim}{\varphi} \cdot \underset{\approx}{P} \cdot d\underset{\sim}{\varphi} + g^2 d\tau^2]^{\frac{1}{2}} \end{aligned} \right\} \tag{A15}$$

to yield

$$z_s = z_t - z_b = \int_0^{t-b=s} \frac{1}{f(\zeta(b) + \zeta(\tau'))} [\dot{\underset{\sim}{\varphi}}^b \cdot \underset{\approx}{P} \cdot \dot{\underset{\sim}{\varphi}}^b + g^2]^{\frac{1}{2}} d\tau' \tag{A16}$$

and

$$z = \int_0^t \frac{1}{f(\zeta(\tau))} [\dot{\underset{\sim}{\varphi}}^b \cdot \underset{\approx}{P} \cdot \dot{\underset{\sim}{\varphi}}^b + g^2]^{\frac{1}{2}} d\tau . \tag{A17}$$

Comparison of (A17) and (A16) yields

$$z_s = z \text{ for } t = s \text{ if}$$

$$f(\zeta(\tau)) = f(\zeta(b) + \zeta(\tau')) . \tag{A18}$$

Again condition (A18) is true if f is introduced as a constant.

We can conclude that the introduction of an intrinsic time (A7) alone does not quarantee a multiple convolution integral series in $z - z'$ to represent history dependence in the sense of plasticity. The function f introduced in (A6) and (A15) plays a crucial role. Only if f is not a constant can stimulus induced variable heredity or history dependence in the sense of plasticity be reproduced. This observation coincides with the findings in [23], p.537 where it is shown that the choice of f = constant precludes the modelling of cross

A comparison of (A12) and (A13) shows that $z_s = z$ for $t = s$ if

$$f(\xi(a) + \xi(\tau)) = f(\xi(\tau)), \tag{A14}$$

since the name of the dummy variable is immaterial. Condition (A14) is certainly true if f is a constant.

The same procedure can be repeated for the "old endochronic" time proposed in [22,23] where

$$\left.\begin{aligned} dz &= \frac{1}{f(\zeta)} \frac{d\zeta}{d\tau} d\tau \\ \text{with} \qquad d\zeta &= [d\underset{\sim}{\varphi} \cdot \underset{\approx}{P} \cdot d\underset{\sim}{\varphi} + g^2 d\tau^2]^{\frac{1}{2}} \end{aligned}\right\} \tag{A15}$$

to yield

$$z_s = z_t - z_b = \int_0^{t-b=s} \frac{1}{f(\zeta(b) + \zeta(\tau'))} [\dot{\underset{\sim}{\varphi}}^b \cdot \underset{\approx}{P} \cdot \dot{\underset{\sim}{\varphi}}^b + g^2]^{\frac{1}{2}} d\tau' \tag{A16}$$

and

$$z = \int_0^t \frac{1}{f(\zeta(\tau))} [\dot{\underset{\sim}{\varphi}}^b \cdot \underset{\approx}{P} \cdot \dot{\underset{\sim}{\varphi}}^b + g^2]^{\frac{1}{2}} d\tau . \tag{A17}$$

Comparison of (A17) and (A16) yields

$$z_s = z \text{ for } t = s \text{ if}$$

$$f(\zeta(\tau)) = f(\zeta(b) + \zeta(\tau')) . \tag{A18}$$

Again condition (A18) is true if f is introduced as a constant.

We can conclude that the introduction of an intrinsic time (A7) alone does not quarantee a multiple convolution integral series in $z - z'$ to represent history dependence in the sense of plasticity. The function f introduced in (A6) and (A15) plays a crucial role. Only if f is not a constant can stimulus induced variable heredity or history dependence in the sense of plasticity be reproduced. This observation coincides with the findings in [23], p.537 where it is shown that the choice of f = constant precludes the modelling of cross

hardening, a true plasticity effect. However, we do not see at all a reason for including the material tensor $\underset{\sim}{P}$ in (A7) or (A15). It can be set to unity without any influence on the comparison between (A16) and (A17) or (A12) and (A13).

Further note that $1/f(\zeta)$ multiplies the expression in square brackets in (A16). Consider now the rate-independent case in (A16), i.e., $g = 0$. To reproduce plasticity f should not be a constant. Suppose $f = f(\zeta)$. Now consider $\underset{\sim}{P} = \underset{\sim}{Q}$ but $g \neq 0$, then the expressions (A16) and (A17) for z and z_s, respectively will differ. As a consequence the convolution integral series will represent aging in real time, an unacceptable result. The new intrinsic time (A5) does not suffer from this difficulty, see the discussion on p.860 of [29].

It is further of interest that $f = f(\xi)$ in (A8) or $f = f(\zeta)$ in (A16) will make the convolution integral series represent stimulus induced variable heredity and not stimulus induced variable hereditary response as required by our *necessary condition*. Plasticity effects are introduced for every loading. Elasticity is then excluded from such representations.

Since the function f is absent in [24], see (52) - (54) of [24], the functionals employed there represent invariable heredity. History dependence in the sense of plasticity cannot be represented by this theory of "Thermo-Plastic Materials with Memory".

APPENDIX II

HYPOELASTICITY REPRESENTS INVARIABLE HEREDITY

Using (28) we set $\underset{\sim}{D} = \underset{\sim}{\varphi}$ and $\underset{\sim}{T} = \underset{\sim}{\rho}$. From (7) and Fig.1, specimen II, $\underset{\sim}{\rho}^{II}$ is defined as

$$\underset{\sim}{\rho}^{II}(\eta) = \underset{\sim}{\rho}(\eta + b) - \underset{\sim}{\rho}(b) . \tag{B1}$$

For specimen I $\underset{\sim}{\rho}^{I}$ is obtained as the solution of

$$\dot{\underset{\sim}{\rho}}^{I} = \underset{\approx}{f}(\underset{\sim}{\rho}^{I}) \cdot \underset{\sim}{\varphi}^{b} \tag{B2}$$

subject to $\underset{\sim}{\rho}^{I}(0) = \underset{\sim}{0}$.

For specimen II we get

$$\dot{\underset{\sim}{\rho}}(\tau) = \underset{\approx}{f}(\underset{\sim}{\rho}(\tau)) \cdot {}^{a}\underset{\sim}{\varphi}^{b}(\tau - b) . \tag{B3}$$

Setting $\tau = \eta + b$ (B3) transforms to

$$\dot{\underset{\sim}{\rho}}(\eta + b) = \underset{\approx}{f}(\underset{\sim}{\rho}(\eta + b)) \cdot {}^{a}\underset{\sim}{\varphi}^{b}(\eta) . \tag{B4}$$

To obtain $\underset{\sim}{\rho}^{II}$ we must form the difference defined in (B1) and use it in (B4), i.e.

$$\dot{\underset{\sim}{\rho}}(\eta + b) - \dot{\underset{\sim}{\rho}}(b) = \underset{\approx}{f}(\underset{\sim}{\rho}(\eta + b) - \underset{\sim}{\rho}(b)) \cdot {}^{a}\underset{\sim}{\varphi}^{b}(\eta)$$

$$\dot{\underset{\sim}{\rho}}^{II}(\eta) = \underset{\approx}{f}(\underset{\sim}{\rho}^{II}(\eta)) \cdot {}^{a}\underset{\sim}{\varphi}^{b}(\eta) \tag{B5}$$

subject to $\underset{\sim}{\rho}^{II}(0) = \underset{\sim}{0}$.

Since $\underset{\sim}{\varphi}^{b} = {}^{a}\underset{\sim}{\varphi}^{b}$ comparison of (B2) and (B5) and of the initial conditions shows that

$$\underset{\sim}{\rho}^{I} = \underset{\sim}{\rho}^{II} \tag{B6}$$

always. Hypoelasticity represents therefore invariable heredity.

The introduction of the new origin defined by (B1) is very important here. Without it the solution of (B4) would nonlinearly depend on the initial condition $\underset{\sim}{\rho}(b)$.